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M.A./M.Sc. (Sem. I) Examination, Dec. 2022

MATHEMATICS

Paper MATH 1 C(1)

(Algebra—I)

Time Allowed : Three Hours

Max. Marks : 100

*This question paper contains **three** sections as under :*

Section-A **Max. Marks-10**

*This section contains **one** compulsory question with **10** parts, having **2** parts from each unit, short answer in **20** words for each part. All questions carry equal marks.*

Section-B **Max. Marks-50**

*This section contains **10** questions having **2** questions from each **unit**. Answer **5** questions (**250 words each**) selecting **one** question from each unit. All questions carry equal marks.*

Section-C **Max. Marks-40**

*This section contains **4** descriptive type questions (question may have sub-division) covering all units but not more than **one** question from each unit. Answer any **two** questions. (**500** words **each**). All questions carry equal marks.*

Section A

1. (a) Define **kernel** of linear transformation.

   (b) Define characteristic value problem.

   (c) Define Bilinear form.

   (d) Define Hermitian form.

(e) Give the definition of Inner product space.

(f) Define Homomorphism of groups.

(g) Define subnormal series.

(h) Define Euclidean ring.

(i) Define Unique Factorisation Domain.

(j) Define Field Extension.

Section B

UNIT-I

2. Prove that the kernel of a linear transformation is a subspace.

Or

Show that the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. Hence find its inverse matrix.

UNIT-II

3. Let V(F) and V'(F) be the vector spaces having bases as $B = \{b_1, b_2, \ldots\ldots, b_m\}$ and $B' = \{b_1', b_2', \ldots\ldots, b_n'\}$ respectively. The mapping ψ which associates each bilinear form on $V \times V'$ to its matrix relative to bases B and B' is an isomorphism of the vector space of all bilinear forms on $V \times V'$ to the vector space $M_{m\times n}$ of $n \times n$ matrices of F.

i.e. $H(V, V', F) \cong M_{m\times n}$.

Or

If the field F of characteristic $\neq 2$, then every symmetric bilinear form on V(F) is uniquely determined by the corresponding Quadratic form. Prove.

UNIT-III

Prove that if $\alpha, \beta \in V$, then $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.

Or

Show that if G is a group with $N \Delta G$, and if $f : G \to \frac{G}{N}$ is such that :

$$f(x) = N_x \ \forall x \in G.$$

Then f is a homomorphism of G onto $\frac{G}{N}$ and Ker (f) = N.

UNIT-IV

5. If O(G) = P^n; where P is a prime number, then its centre $z \neq \{e\}$ i.e. $O[Z(G)] > 1$.

Or

Prove that an infinite abelian group does not have a composition series.

[Contd....

UNIT-V

6. Show that if $a \in K$ be a root of $p(x) \in F(x)$ where $F \subseteq K$. Then in K $[x]$, $(x - a)$ is a divisor of $p(x)$ i.e. $(x - a)/p(x)$.

Or

If F be any subfield of K and G be any group of automorphism to K. Then the fixed field of G is a subfield of K.

Section C

7. If H and K be two subgroups of a group G such that $H \Delta G$, then show that :

(i) $H \cap K \Delta K$

(ii) $\frac{K}{H \cap K} \cong \frac{HK}{H}$.

8. Prove that any *two* composition series of a finite group G are equivalent i.e. are of same length and have isomorphic factors.

9. (a) If $t : V(F) \rightarrow V'(F)$ is a linear transformation where $V(F)$ is a finite dimensional vector space, then show :

$$\text{Rank } (t) + \text{Nullity } (t) = \text{Dim. } (V)$$

(b) Show that all the eigen values of a Hermitian matrix are real.

10. (a) Reduce the QF $A(x, x) = X^T AX$ to canonical form; where :

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix}$$

(b) If $\alpha, \beta \in V(F)$, then $|(\alpha, \beta)| \leq \|\alpha\| \cdot \|\beta\|$, prove it.